Differential Geometry I: Curves and surfaces

13.1 Recall that, in exercise 10.7 we constructed the catenoid as the surface of revolution of the catenary curve:

$$\alpha(t) = (t, \cosh(t)).$$

In particular, the catenoid surface can be parametrized as follows (as a surface of revolution around the Ox-axis):

$$\psi(s,\theta) = \big(\log(s+\sqrt{1+s^2}),\ \sqrt{1+s^2}\cos\theta,\ \sqrt{1+s^2}\sin\theta\big).$$

Compute the metric tensor, the second fundamental form and the Weingarten map of the catenoid. What is its Gaussian curvature? Show that it's also a minimal surface, i.e. that its mean curvature vanishes.

Remark. The catenoid is the surface of minimal area with boundary given by two rings of the same radius which are sufficiently close to each other (think of a surface made of soap bubble connecting the two rings). When the rings are sufficiently far apart, however, the surface with this boundary which is of minimal area is instead the union of the two disjoint flat disks bounded by the rings.

Solution. The adapted frame for the chosen parametrization is given by

$$\begin{cases} b_1 = \dfrac{\partial\psi}{\partial s} = \dfrac{(1, s\cos\theta, s\sin\theta)}{\sqrt{1+s^2}}, \\ b_2 = \dfrac{\partial\psi}{\partial\theta} = \sqrt{1+s^2}\,(0, -\sin\theta, \cos\theta), \\ n = \dfrac{b_1\times b_2}{\|b_1\times b_2\|} = \dfrac{(s, -\cos\theta, -\sin\theta)}{\sqrt{1+s^2}}. \end{cases}$$

The matrix of the metric tensor is therefore

$$G(s,\theta) = \begin{pmatrix} \langle b_1, b_1\rangle & \langle b_1, b_2\rangle \\ \langle b_2, b_1\rangle & \langle b_2, b_2\rangle \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1+s^2 \end{pmatrix}.$$

The coefficients of the second fundamental form can be computed in several ways:

$$\begin{cases} h_{11} = \left\langle \dfrac{\partial b_1}{\partial s}, n\right\rangle = -\left\langle b_1, \dfrac{\partial n}{\partial s}\right\rangle, \\ h_{12} = h_{21} = \left\langle \dfrac{\partial b_1}{\partial \theta}, n\right\rangle = -\left\langle b_1, \dfrac{\partial n}{\partial \theta}\right\rangle = -\left\langle b_2, \dfrac{\partial n}{\partial s}\right\rangle, \\ h_{22} = \left\langle \dfrac{\partial b_2}{\partial \theta}, n\right\rangle = -\left\langle b_2, \dfrac{\partial n}{\partial \theta}\right\rangle. \end{cases}$$

We have

$$\frac{\partial n}{\partial s} = \frac{(1, s\cos\theta, s\sin\theta)}{(1+s^2)^{3/2}}, \qquad \frac{\partial n}{\partial\theta} = \frac{(0, \sin\theta, -\cos\theta)}{\sqrt{1+s^2}}.$$

Thus

$$h_{11} = -\left\langle b_1, \frac{\partial n}{\partial s} \right\rangle = -\frac{1}{1+s^2}, \quad h_{22} = -\left\langle b_2, \frac{\partial n}{\partial \theta} \right\rangle = 1, \quad h_{12} = 0.$$

The matrix of the second fundamental form is therefore

$$H(s,\theta) = \begin{pmatrix} -\frac{1}{1+s^2} & 0 \\ 0 & 1 \end{pmatrix}.$$

The Weingarten matrix can now be computed using the formula

$$L = -G^{-1}H = \frac{1}{1+s^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In this example, it is not difficult to compute this matrix directly. The Weingarten map L is the differential of the normal vector, expressed in the basis $\{b_1, b_2\}$ of the tangent plane to the catenoid. Here the computations are simple:

$$L(b_1) = \frac{\partial n}{\partial s} = \frac{1}{1+s^2} b_1, \qquad L(b_2) = \frac{\partial n}{\partial \theta} = -\frac{1}{1+s^2} b_2.$$

Thus the matrix of L in the basis $\{b_1, b_2\}$ is

$$L = \frac{1}{1+s^2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now obtain the Gaussian curvature and the mean curvature of the catenoid:

$$K = \det(L) = -\frac{1}{(1+s^2)^2}, \qquad H = -\frac{1}{2} \operatorname{Tr}(L) = 0.$$

(as usual, we use the same symbol H for the matrix of the second fundamental form and the mean curvature).

Remarks.

(i) One can also compute the Gaussian curvature directly from the matrices G and H using

$$K = \frac{\det(H)}{\det(G)} = -\frac{1}{(1+s^2)^2}.$$

(ii) The catenoid is a surface with zero mean curvature. Such a surface is called a *minimal surface*.

13.2 Let $S \subset \mathbb{R}^3$ be a C^2 surface, and denote by H and K its mean and Gaussian curvature respectively.

(a) Show that the principal curvatures are given by

$$k_1,\ k_2 \ =\ H \pm \sqrt{H^2 - K}.$$

(b) Show that if S is a *minimal* surface, then its Gaussian curvature satisfies $K \leqslant 0$.

Solution. (i) This is elementary algebra. From $H = \frac{1}{2}(k_1 + k_2)$ and $K = k_1 k_2$ we deduce

$$H^2 - K = \frac{1}{4}(k_1 - k_2)^2.$$

Assume (possibly after exchanging k_1 and k_2) that $k_1 \geqslant k_2$. Then

$$\begin{cases} \frac{1}{2}(k_1 + k_2) = H, \\ \frac{1}{2}(k_1 - k_2) = \sqrt{H^2 - K}. \end{cases}$$

Thus

$$k_1 = H + \sqrt{H^2 - K}, \qquad k_2 = H - \sqrt{H^2 - K}.$$

(ii) If S is a minimal surface, we have $H = 0 \rightarrow= k_1 + k_2 = 0$. Therefore, $K = k_1 k_2 = k_1(-k_1) = -k_1^2 \leqslant 0$.

13.3 Show that the Gaussian curvature and the mean curvature can be expressed in terms of the coefficients (g_{ij}) and (h_{ij}) of the first and second fundamental forms by

$$K = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2}, \qquad H = \frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{2\,(g_{11}g_{22} - g_{12}^2)}.$$

Solution. This is matrix calculus. We have $L = -G^{-1}H$ with

$$G = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}, \qquad H = \begin{pmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{pmatrix}.$$

The first identity is simply the relation

$$K = \det(L) = \frac{\det(H)}{\det(G)}.$$

For the second identity, we compute

$$L = -G^{-1}H = \frac{1}{g_{11}g_{22} - g_{12}^2} \begin{pmatrix} -g_{22}h_{11} + g_{12}h_{12} & -g_{22}h_{12} + g_{12}h_{22} \\ g_{12}h_{11} - g_{11}h_{12} & g_{12}h_{12} - g_{11}h_{22} \end{pmatrix},$$

and therefore

$$H = -\frac{1}{2}\,\mathrm{Tr}(L) = \frac{1}{2}\frac{g_{11}h_{22} - 2g_{12}h_{12} + g_{22}h_{11}}{g_{11}g_{22} - g_{12}^2}.$$

13.4 Let $\psi_1 : \Omega \to S \subset \mathbb{R}^3$ be a surface of class C^2 and let $\lambda > 0$. Define $\psi_2 = \lambda\psi_1 : \Omega \to \lambda S \subset \mathbb{R}^3$, the surface obtained by applying a homothety of ratio λ. What is the relation between the Gaussian curvature $K_1(u, v)$ at a point $p = \psi_1(u, v) \in S$ and the Gaussian curvature $K_2(u, v)$ at the point $q = \lambda p = \psi_2(u, v) \in \lambda S$?

Solution. The curvature of any curve drawn on the surface S is multiplied by $1/\lambda$ under the homothety of ratio λ. The Gaussian curvature is the product of the two principal curvatures, and is therefore multiplied by $1/\lambda^2$.

An alternative way of obtaining the same result: For the local parametrizations ψ_1 and $\psi_1' = \lambda\psi_1$ (let us use ψ_1' in place of ψ_2, the corresponding adapted frames $\{b_1, b_2, n\} = \{\frac{\partial\psi_1}{\partial u_1}, \frac{\partial\psi_1}{\partial u_2}, \frac{b_1\times b_2}{\|b_1\times b_2\|}$ and $\{b_1', b_2', n'\}$ are related by $b_1' = \lambda b_1$, $b_2' = \lambda b_2$ and $n' = n$ (this is easy to verify). Therefore

$$g_{ij}' = \langle b_i', b_j'\rangle = \lambda^2\langle b_i, b_j\rangle = \lambda^2 g_{ij}$$

and

$$h_{ij}' = \langle \frac{\partial b_i'}{\partial u_j}, n'\rangle = \lambda\langle \frac{\partial b_i}{\partial u_j}, n\rangle = \lambda h_{ij}.$$

Therefore, denoting with H the matrix of the second fundamental form:

$$K' = \frac{\det(H')}{\det(G')} = \frac{\det(\lambda H)}{\det(\lambda^2 G)} = \frac{\lambda^2\det(H)}{\lambda^4\det(G)} = \frac{1}{\lambda^2}\frac{\det(H)}{\det(G)} = \frac{1}{\lambda^2}K.$$

13.5 A regular C^2 curve γ on a surface S is a *line of curvature* if its normal curvature is everywhere a principal curvature. Show that γ is a line of curvature if and only if its geodesic torsion is zero.

Solution. The second Darboux equation states that

$$\frac{1}{V}\dot{n} = -k_n T + \tau_g\mu.$$

We have written $n(t) = n(\gamma(t))$ for simplicity, so expanding gives

$$V(-k_n T + \tau_g\mu) = \dot{n} = \frac{d}{dt}n(\gamma(t)) = dn(\dot{\gamma}(t)) = -L(\dot{\gamma}(t)).$$

Thus

$$L(\dot{\gamma}(t)) = V(-k_n T + \tau_g\mu) = k_n\dot{\gamma} + V\tau_g\mu,$$

which means that $\dot{\gamma}(t)$ is an eigenvector of the Weingarten map if and only if $\tau_g = 0$.

13.6 Prove that the catenoid (see Ex. 13.1) and the helicoid (see Ex. 10.6) are locally isometric (Hint: Consider the expression of the metric tensor in the parametrizations we have already considered). Are they globally isometric?

Remark. The fact that the catenoid and the helicoid are locally isometric implies that these two surfaces have the same Gaussian curvature at the corresponding points matched by the local isometry, as a corollary of the *Theorema Egregium*. If you want, you can try to confirm this by computation.

Solution. The metric tensor of the helicoid $\mathcal{H}$ in the parameters $(u, v) \in \mathbb{R}^2$ is

$$\begin{pmatrix} 1+v^2 & 0 \\ 0 & 1 \end{pmatrix},$$

and the metric tensor of the catenoid $\mathcal{C}$ in the parameters $(s, \theta) \in \mathbb{R} \times [0, 2\pi)$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1+s^2 \end{pmatrix}.$$

The change of variables $u = \theta + u_0$, $v = s$ therefore defines a local isometry between the two surfaces for any $u_0 \in \mathbb{R}$.

If we would like to be more pedantic: For the above representations $\psi_1 : \Omega_1 = \mathbb{R}^2 \to \mathcal{H}$ and $\psi_2 : \Omega_2 = \mathbb{R} \times [0, 2\pi) \to \mathcal{C}$ of the surfaces, the map $F : \mathcal{C} \to \mathcal{H}$ defined by $F = \psi_1 \circ F' \circ \psi_2^{-1}$, where $F' : \Omega_2 \to \Omega_1$ is defined by $F'(s, \theta) = (\theta + u_0, s)$, is a local isometry. This is because the metric tensors satisfy $G_2(s, \theta) = dF^T(s, \theta) \cdot G_1(F'(s, \theta)) \cdot dF(s, \theta)$. Of course, you don't have to be so pedantic for the exam.

13.6 Let $S \subset \mathbb{R}^3$ be a surface of class C^2 and $\gamma : I \to S$ be a C^2 regular curve. We will say that γ is an *asymptotic* curve of S if the curvature vector $K_\gamma(t)$ of γ is tangent to S for all $t \in I$.

(a) Show that the following conditions are equivalent (for the last one, you can assume in addition that γ is a biregular curve):

1. γ is an asymptotic curve of S,
2. The normal curvature k_n of γ vanishes everywhere,
3. $h(\dot{\gamma}, \dot{\gamma}) = 0$ everywhere on γ,
4. The binormal of γ is parallel to the normal n of the surface.

(b) Let $\gamma : I \to S$ be an asymptotic curve of S. Prove that, for any $t \in I$, the Gauss curvature $K(\gamma(t))$ of S at $\gamma(t)$ satisfies $K(\gamma(t)) \leqslant 0$.

(c) Prove that, for a ruled surface, the line segments (corresponding to the ruling) are asymptotic curves (you don't really have to do a computation).

Solution. (a) We need to show that the four conditions are equivalent (we will label them as 1,2,3 and 4, in the order they are stated). We first prove the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Then, under the assumption of regularity in the Frenet sense, we prove the implications (1) $\Leftrightarrow$ (4).

(1) ⇔ (2) The normal curvature of γ is defined by

$$k_n(t) = \langle K_\gamma(t), n(t)\rangle.$$

Thus $K_\gamma(t) \perp n(t)$ if and only if $k_n(t) = 0$.

(2) ⇔ (3) This equivalence follows from the following formula, seen in the course, which states that the normal curvature of γ is given by

$$k_n(t) = \frac{h(\dot\gamma(t), \dot\gamma(t))}{\|\dot\gamma(t)\|^2}.$$

Let us recall the proof of this formula. We have

$$\langle n(\gamma(t)), \dot\gamma(t)\rangle = 0 \quad \text{for all } t \in I,$$

thus

$$\left\langle \frac{d}{dt} n(\gamma(t)), \dot\gamma(t) \right\rangle + \langle n(\gamma(t)), \ddot\gamma(t)\rangle = 0.$$

Using the acceleration formula

$$\ddot\gamma(t) = V^2(t) K_\gamma(t) + \dot V(t) T_\gamma(t),$$

we see that

$$\langle n(\gamma(t)), \ddot\gamma(t)\rangle = V^2(t)\langle n(\gamma(t)), K_\gamma(t)\rangle + \dot V(t)\langle n(\gamma(t)), T_\gamma(t)\rangle.$$

The last term is zero, hence

$$\langle n(\gamma(t)), \ddot\gamma(t)\rangle = V^2(t)\langle n(\gamma(t)), K_\gamma(t)\rangle = \|\dot\gamma(t)\|^2 k_n(t).$$

On the other hand,

$$\left\langle \frac{d}{dt} n(\gamma(t)), \dot\gamma(t) \right\rangle = \langle dn(\gamma(t)), \dot\gamma(t)\rangle = \langle L(\dot\gamma(t)), \dot\gamma(t)\rangle = -h(\dot\gamma(t), \dot\gamma(t)).$$

From the three previous identities, we deduce that

$$\|\dot\gamma(t)\|^2 k_n(t) = h(\dot\gamma(t), \dot\gamma(t)).$$

Thus we have shown (1) ⇔ (2) ⇔ (3) for any C^2 curve on the surface S.

We now assume that the curve γ is regular in the Frenet sense, and we denote by $\{T, N, B\}$ the Frenet frame. Under this hypothesis we show (1) ⇒ (4).

(1) ⇒ (4) By hypothesis, $N = \frac{1}{\kappa} K$ is tangent to the surface S. Consequently $N = \pm\mu$, where $\mu = n \times T$ is the corresponding of the Darboux frame. We therefore have

$$B = T \times N = \pm T \times \mu = \pm n.$$

(4) ⇒ (1) If $B = \pm n$, then, since K is parallel to N and $N \perp B$, we have that $K \perp n$. Therefore, K is tanfent to the surface.

(b) Let $p = \gamma(t)$. Since γ is an asymptotic curve, from the previous part we have that

$$h(\dot{\gamma}, \dot{\gamma}) = 0.$$

In the orthonormal frame $\{T, \mu\}$ of T_pS (recall that T is parallel to $\dot{\gamma}$), the above means that the matrix H of the second fundamental form takes the form

$$H = \begin{pmatrix} 0 & h_{12} \\ h_{12} & h_{22} \end{pmatrix}.$$

Therefore, $\det(H) = -h_{12}^2 \leqslant 0$. On the other hand, since, with respect to this frame, the metric tensor G is the identity, we have that $\det(G) = 1$ (note that the metric tensor is always positive definite; that's an easy consequence of the Cauchy-Schwarz inequality). Therefore,

$$K = \frac{\det(H)}{\det(G)} \leqslant 0.$$

(c) For a ruled surface, the curves corresponding to the ruling (straight lines) have zero curvature vector; in particular, the curvature vector is tangent to the surface.

13.7 Let $\gamma : I \to \mathbb{R}^3$ be a C^3 biregular curve. Assume that $\|\dot{\gamma}\|$ is constant.

(a) Prove that γ is a geodesic of the ruled surface

$$S : \quad \psi(u, v) = \gamma(u) + v\, B_\gamma(u),$$

where $B_\gamma(u)$ is the binormal vector of γ.

(b) Prove that γ is an asymptotic of the ruled surface

$$S : \quad \psi(u, v) = \gamma(u) + v\, N_\gamma(u),$$

where $N_\gamma(u)$ is the principal normal of γ.

Solution. (a) By construction, the vector μ is (up to sign) the vector orthogonal to $\dot{\gamma}$ and tangent to the surface. Thus $\mu = \pm B$ for the ruled surface S of this exercise. Comparing the Darboux equations with the Serret–Frenet equations, we see that the geodesic curvature k_g of γ is zero. We conclude by recalling that a curve on a surface is a geodesic if and only if it has constant speed and zero geodesic curvature.

Another argument. Let $S = \psi(I \times (-\varepsilon, \varepsilon)) \subset \mathbb{R}^3$ be the surface parametrized by ψ (for $\varepsilon > 0$ small enough so that the surface is regular). Let $n : S \to S^2$ be the co-orientation of the surface and let N_γ be the principal normal vector of γ.

It is clear that for any $u \in I$, the vectors $\dot{\gamma}(u)$ and $B_\gamma(u)$ form a basis of T_pS (with $p = \gamma(u) = \psi(u, 0)$). Consequently, $n = \pm N_\gamma$, and therefore

$$\ddot{\gamma}(u) = V^2 \kappa_\gamma(u) N_\gamma(u) = \pm V^2 \kappa_\gamma(u) n(\psi(u, 0)),$$

where $V = \|\dot{\gamma}\|$. The acceleration is colinear with the normal vector to the surface, which means that γ is a geodesic.

(b) In this case, at the point $p = \gamma(u)$ of S, the tangent space T_pS is spanned by $\dot{\gamma}(u)$ and $N_\gamma(u)$. Since the curvature vector K_γ of γ is always parallel to the principal normal N_γ, we infer that K_γ is tangent to the surface, hence γ is an asymptotic curve.

13.8 **(a)** First, let us recall some notions from linear algebra: For any symmetric matrix $A \in \mathcal{M}_n(\mathbb{R})$, we say that $A \geqslant 0$ if, for every $v \in \mathbb{R}^n$, we have

$$v^T A v \geqslant 0.$$

We also say that $A \geqslant B$ iff $A - B \geqslant 0$. Prove that, if $A \geqslant B$, then $\mathrm{Tr}(A) \geqslant \mathrm{Tr}(B)$.

(b) Let $\Omega \subseteq \mathbb{R}^2$ be a domain containing 0 and let $f_1, f_2 : \Omega \to \mathbb{R}$ be a pair of C^2 functions satisfying:

$$f_1(0) = f_2(0) = 0, \quad \nabla f_1(0) = \nabla f_2(0) = 0$$

and

$$f_1(x, y) \leqslant f_2(x, y) \quad \text{for all } (x, y) \in \Omega.$$

Let us denote by $S_1, S_2 \subset \mathbb{R}^3$ the graphs $\{z = f_1(x, y)\}$ and $\{z = f_2(x, y)\}$, respectively; note that S_1 and S_2 "touch" at 0. Let us fix the coorientations n_1, n_2 of S_1, S_2 so that, at 0, we have $n_1 = n_2 = (0, 0, 1)$. Show that the mean curvatures H_1, H_2 satisfy

$$H_1(0) \leqslant H_2(0).$$

Hint: You might want to first express the second fundamental forms of S_i at 0 in terms of $\nabla^2 f_i(0)$.

(c*) Show that there is no *compact* minimal surface in $\mathbb{R}^3$.

Hint: Argue by contradiction; assuming that S is a compact minimal surface, consider the smallest sphere S_R containing S in its interior. Such a sphere must touch S at a point p; apply part (b) for the surfaces S and S_R around that point.

Solution. (a) Note that, if A is a matrix, then the diagonal A_{ii} element of A is equal to $\langle Ae_i, e_i \rangle$, where e_i is the i-th canonical basis element of $\mathbb{R}^n$. Therefore,

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii} = \sum_{i=1}^{n} \langle Ae_i, e_i \rangle.$$

If $A \geqslant B$, then we have $\langle (A-B)e_i, e_i\rangle \geqslant 0$ for all $i = 1, \dots, n$ and, therefore

$$\mathrm{Tr}(A-B) = \sum_{i=1}^{n} \langle (A-B)e_i, e_i\rangle \geqslant 0,$$

which gives the required result.

(b) Since $f_1(0) = f_2(0) = 0$ and $\nabla f_1(0) = \nabla f_2(0) = 0$, the Taylor expansion of the functions f_1, f_2 around 0 reads:

$$f_i(x,y) = \frac{1}{2}\begin{bmatrix} x & y\end{bmatrix} \nabla^2 f_i(0) \begin{bmatrix} x \\ y\end{bmatrix} + o(x^2+y^2),$$

where we use the matrix notation for the Hessian:

$$\nabla^2 f_i(0) = \begin{bmatrix} \partial^2_{xx} f_i(0) & \partial^2_{xy} f_i(0) \\ \partial^2_{xy} f_i(0) & \partial^2_{yy} f_i(0)\end{bmatrix}.$$

For any $(v,w) \in \mathbb{R}^2$ and any $\lambda > 0$, if we set $(x,y) = (\lambda v, \lambda w)$ in the above, we get

$$\frac{1}{\lambda^2} f_i(x,y) = \frac{1}{2}\begin{bmatrix} v & w\end{bmatrix} \nabla^2 f_i(0) \begin{bmatrix} v \\ w\end{bmatrix} + o_\lambda(1)(v^2+w^2).$$

Using the assumption that $f_1(x,y) \leqslant f_2(x,y)$, we infer that

$$\begin{aligned} &\frac{1}{\lambda^2} f_1(x,y) \leqslant \frac{1}{\lambda^2} f_2(x,y) \\ &\quad \Rightarrow \quad \frac{1}{2}\begin{bmatrix} v & w\end{bmatrix} \nabla^2 f_1(0) \begin{bmatrix} v \\ w\end{bmatrix} + o_\lambda(1)(v^2+w^2) \leqslant \frac{1}{2}\begin{bmatrix} v & w\end{bmatrix} \nabla^2 f_2(0) \begin{bmatrix} v \\ w\end{bmatrix} + o_\lambda(1)(v^2+w^2) \end{aligned}$$

and, therefore, after sending $\lambda \to 0$ (so that the terms $o_\lambda(1)$ go to 0), we infer that

$$\begin{bmatrix} v & w\end{bmatrix} \nabla^2 f_1(0) \begin{bmatrix} v \\ w\end{bmatrix} \leqslant \begin{bmatrix} v & w\end{bmatrix} \nabla^2 f_2(0) \begin{bmatrix} v \\ w\end{bmatrix}.$$

In particular, the matrices $\nabla^2 f_i(0)$ satisfy

$$\nabla^2 f_1(0) \leqslant \nabla^2 f_2(0).$$

Let us now consider the surfaces S_i, $i = 1, 2$, which are the graphs of the functions f_i. Since $f_1(0) = f_2(0) = 0$ and $\nabla f_1(0) = \nabla f_2(0) = 0$, those surfaces both contain the origin 0 and are tangent there, with tangent space

$$T_0 S_1 = T_0 S_2 = \{z = 0\}.$$

Let us pick a coorientation for those surfaces so that, at the point 0,

$$n_1(0) = n_2(0) = e_3$$

(where $e_3 = (0,0,1)$). With respect to the parametrization $(x,y) \to (x, y, f_i(x,y))$ of those surfaces, the associated basis $\{b_1(0), b_2(0)\}$ at 0 for both surfaces is $\{e_1, e_2\}$. As we computed in the 11^{th} week of the course, the matrix of the second fundamental form of each graph S_i at 0 is then simply given by

$$\mathbb{H}_i(0) = \nabla^2 f_i(0).$$

Therefore,

$$\mathbb{H}_1(0) \leqslant \mathbb{H}_2(0).$$

Note also that the associated basis $\{b_1, b_2\}$ of the tangent space of S_1 (corresponding to the parametrization $(x, y) \to (x, y, f(x, y))$ becomes

$$\{b_1(0), b_2(0)\} = \{e_1, e_2\}$$

(and similarly for S_2. Therefore, the metric tensors satisfy

$$G_1(0) = G_2(0) = \text{Id}.$$

As a result, the matrices of the shape operators $L_i = -G_i^{-1}\mathbb{H}_i$ satisfy $L_i(0) = -\mathbb{H}_i(0)$ and hence:

$$L_1(0) \geqslant L_2(0)$$

As a result, the mean curvatures $H_i = -\frac{1}{2}\text{Tr}(L_i)$ satisfy

$$H_1(0) \leqslant H_2(0).$$

(c) Let S be a compact C^2 minimal surface in $\mathbb{R}^3$. Let $q \in \mathbb{R}^3 \setminus S$ and let $r : \mathbb{R}^3 \to [0, +\infty)$, $r(x) = \|x - q\|$ be the distance function from q. Since S is compact, the function r attains a (non-zero) maximum on S, say at the point $p \in S$; that is to say, $r(z) \leqslant r(p) \doteq R$ for all $z \in S$. Let $S_R = \big\{x \in \mathbb{R}^3 : r(x) = R\big\}$ be the sphere of radius R centered at q and $B_R = \big\{x \in \mathbb{R}^3 : r(x) = R\big\}$ be the corresponding (closed) ball. Then, the previous inequality is equivalent to the fact that $S \subset B_R$. Since $p \in S$ and $p \in S_R$, this means that S and S_R have the same tangent plane at p (i.e. they are tangent): if that was not the case, namely that T_pS and T_pS_R were intersecting transversally, then there would be a tangent vector $v \in T_pS$ which would point to the exterior of B_R; but then any curve on S going through p in the direction of v would exit B_R, which would contradict the fact that $S \subset B_R$.

Let us pick a new Cartesian coordinate system, centered at p, such that the Cartesian basis vectors $\{e_1, e_2\}$ span $T_pS = T_pS_R$ and e_3 (which is perpendicular to $T_pS = T_pS_R$) points to the interior of B_R. Since both S and S_R are of class C^2, they can be expressed locally around p as graphs of C^2 functions over their common tangent space $\{e_1, e_2\}$, namely as graphs $z = f(x, y)$ and $z = f_R(x, y)$ respectively. Since p belongs to both surfaces, we have $f(0) = f_R(0) = 0$. Since they are both tangent to $\{e_1, e_2\}$, we have $\nabla f(0) = \nabla f_R(0) = 0$. Finally, since S is contained inside B_R, we have $f(x, y) \geqslant f_R(x, y)$ for all (x, y) in a neighborhood of 0. Thus, from the previous part, the mean curvatures of S and S_R, respectively, at p should satisfy

$$H(0) \geqslant H_R(0) = \frac{1}{R}$$

(the last equality following from the computation of the shape operator of the round sphere). But this contradicts our assumption that S is a minimal surface, i.e. that $H = 0$. Hence, S cannot be compact.